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A discrete Lefschetz formula

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Abstract

A Lefschetz formula is given that relates loops in a regular finite graph to traces of a certain representation. As an application the vanishing orders of the Ihara/Bass zeta function are expressed as dimensions of global section spaces of locally constant sheaves.
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0. Introduction

In the nineteen-sixties Yasutaka Ihara [11–14] defined a Selberg-type zeta function for rank one p-adic groups. This zeta function can also be considered as a zeta function attached to a quotient of the Bruhat–Tits building which is a special graph. The theory was later extended by Ki-Ichiro Hashimoto [8–10] to cover bipartite graphs and by Hyman Bass [1] to arbitrary graphs. For a simpler proof and further results see [16,17].

For a finite graph Y this zeta function is defined as

$$Z_Y(u) \stackrel{\text{def}}{=} \prod_p \left(1 - u^{l(p)}\right)^{-1},$$

where the product runs over all prime loops in Y (see Section 2). Bass showed that $Z_Y(u)$ extends to a rational function without zeros, so the inverse, $Z_Y(u)^{-1}$, is a polynomial.

In this paper we use Harmonic Analysis of the automorphism group G of the universal covering tree X of Y to describe the zeta function. We prove a Lefschetz formula (Theorem 5.2) that relates lengths of loops to the trace of co-invariants of the induced representation. This Lefschetz formula gives rise to two descriptions of the poles of the zeta function. Firstly, the poles are given by co-invariants of the regular G-representation given by the fundamental group, and secondly they can be described in terms of a family of locally constant sheaves $\mathcal{F}_\lambda$, $\lambda \in \mathbb{C}$, arising from homogeneous line bundles on the boundary of the tree. The relation is that the pole-order of $Z_Y(u)$ at the point $u = \lambda$ is given by

$$\operatorname{ord}_{u=\lambda} Z_Y(u) = \dim H^0(\mathcal{F}_\lambda).$$

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This assertion may be viewed as a discrete version of the Patterson conjecture [2].

The method of proof is an extension of the Harmonic Analysis of p-adic groups to the automorphism group G. We define the analogues of parabolic groups, split tori, and the like. It turns out that a parabolic P satisfies a Langlands decomposition $P = MAN$, but the group M does not centralize the "split torus" A. Therefore we are forced, when considering "Jacquet modules", to take co-invariants not with respect to the "unipotent radical" N but with respect to the group $S = MN$.

There are two appendices in which generalizations of the theory of Ihara/Bass zeta functions are suggested. It is well known that the zeta functions allow for twists by locally constant sheaves. It is shown in Appendix A that certain constructible sheaves may serve the same purpose. In Appendix B the theory is extended to infinite graphs with simple cusps. This is the type of graphs that occur as quotients of Bruhat–Tits trees of arithmetic groups in positive characteristic [15].

In writing up I did not strive for the most general version of each assertion, simply because it is much easier to read this way. Advanced readers may convince themselves that the theory extends to multigraphs, i.e., where multiple edges and non-regular edges are allowed and that the zeta function may also be twisted by hermitian locally constant sheaves of finite rank, i.e., by finite dimensional unitary representations of the fundamental group. It is, however, unclear to me how far the regularity condition can be relaxed, i.e., whether the results of the paper (or modified versions thereof) hold for general finite graphs.

1. The zeta function

In this paper, a *graph* X consists of a set VX, called the set of *vertices*, together with a set EX of subsets of VX of order two, called *edges*. We say that the edge $e = \{x, y\}$ *connects* the vertices x and y and we also write $e = \overline{x, y}$. We then say that x and y are *adjacent*. The graph is called a *finite graph* if VX is a finite set. For a vertex x the number $val(x) \in \{0, 1, 2, \dots, \infty\}$ of edges ending at x is called the *valency* of x. The graph is called *univalent* or *regular* if the valency is finite and the same for all vertices.

Sometimes we will also consider a graph as a one-dimensional CW-complex, so all edges are identified with the unit interval. It does then make sense to speak of points of the graph and in particular of midpoints of edges. Further we have a path-length metric $d(\cdot, \cdot)$ on X giving each edge the length one.

A (combinatorial) *path* p is a sequence $(v_0, \dots, v_n)$, $n > 0$, of vertices such that v_j and v_{j+1} are adjacent for $j = 0, \dots, n-1$. The number $n = l(p)$ is called the *length* of the path. Likewise, a path can also be described by the oriented edges it passes through. The path is *closed* if $v_0 = v_n$. The path has *no backtracking* if $v_{j-1} \neq v_{j+1}$ for every $j = 1, \dots, n-1$. A closed path is *tail-less* if $v_1 \neq v_{n-1}$. A tail-less closed path with no backtracking is called *reduced*. Two reduced paths $(v_0, \dots, v_n)$ and $(w_0, \dots, w_k)$ are said to be *shift-equivalent* if $k = n$ and there is an $s = 0, 1, 2, \dots$ such that $v_j = w_{j+s}$ where j runs modulo n. A shift equivalence class of reduced paths is called a *loop*. If c is a loop, then $c^2 = cc$ also is a loop and so is any power c^k, $k \in \mathbb{N}$. A loop c is called *prime* if it is not a power d^k of a shorter loop d. Every loop c is a power $c = c_0^{m(c)}$ of a unique prime loop c_0. The number $m(c) \in \mathbb{N}$ is called the *multiplicity* of c. The length $l(c_0)$ is called the *geometric length* of the loop c. Then obviously, $l(c) = m(c)l(c_0)$. Finally, a graph is *connected* if any two vertices can be connected by a path.

Let Y be a finite connected graph. Its *zeta function* is defined by

$$Z_Y(u) \stackrel{\text{def}}{=} \prod_p \left(1 - u^{l(p)}\right)^{-1},$$

where the product runs over all prime loops in Y. It has been shown by H. Bass in [1] that $Z_Y(u)$ always is a rational function, indeed, its inverse $Z_Y(u)^{-1}$ is a polynomial. In this paper we want to study $Z_Y(u)$ of a regular finite graph via the universal covering X of Y.

2. The tree

Let X be a tree, i.e., X is a countable, connected graph that admits no loops. We assume that X is regular. Let $q + 1$ be its valency where $q \in \mathbb{N}$. Let G be the group of automorphisms of X. The group G carries a totally disconnected topology. A basis of neighbourhoods of the unit is given by the compact open subgroups

$$K_E \stackrel{\text{def}}{=} \{g \in G\colon\ ge = e\ \forall e \in E\}$$

where E is an arbitrary nonempty finite set of vertices. Every compact subgroup of G fixes a point x which is either a vertex or the midpoint of an edge. So every maximal compact subgroup is a stabilizer K_x of such a point x. Another way to describe the topology of G is to say that a sequence (g_n) in G converges to $g \in G$ iff for every vertex x there is a natural number $N(x)$ such that for every $n \geqslant N(x)$ we have $g_n(x) = g(x)$.

A *ray* in X is a one sided infinite path without backtracking, so it is a sequence of vertices $(v_0, v_1, \dots)$ such that v_j and v_{j+1} are adjacent and $v_j \neq v_{j+2}$ for $j \geqslant 0$. A *line* is a two-sided infinite path $(\dots, v_{-1}, v_0, v_1, \dots)$ without backtracking.

Two rays $(v_0, v_1, \dots)$ and $(w_0, w_1, \dots)$ are called *parallel* if there are $s \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $v_j = w_{j+s}$ for every $j \geqslant N$. Parallelity is an equivalence relation. The *boundary* ∂X of X is by definition the set of all parallelity classes of rays in X. A boundary point $c \in \partial X$ also is called a *cusp* of X.

The group G acts on the boundary ∂X. A *parabolic subgroup* P of G is the stabilizer of a cusp $b \in \partial X$. We fix a cusp $\infty \in \partial X$ and its stabilizer $P \subset G$. Every vertex x can be joined by a ray to ∞. In other words, given x, there exists a ray $(x = v_0, v_1, \dots)$ which lies in the parallelity class ∞. This ray is unique. We write it as $(v_0(x), v_1(x), \dots)$. Two vertices x, y are said to lie in the same *horosphere* with respect to ∞ if there is $N \in \mathbb{N}$ such that $v_N(x) = v_N(y)$. Then automatically, $v_{N+j}(x) = v_{N+j}(y)$ for $j \geqslant 0$. Loosely speaking a horosphere is the set of all vertices of the same distance to ∞. If $p \in P$ preserves a horosphere, it will preserve every horosphere. Let $S \subset P$ be the subgroup of those elements that preserve horospheres. Let $(v_0, v_1, v_2, \dots)$ be a ray in the class ∞. Let U_j be the group of all $s \in S$ with $sv_j = v_j$. Since any $s \in U_j$ fixes v_j and fixes ∞, it also fixes v_l for $l > j$. So it follows that $U_j \subset U_{j+1}$ and we have

$$S = \bigcup_{j=0}^{\infty} U_j.$$

Note that every U_j is a compact open subgroup of S, so S is the union of its compact open subgroups.

Fix a second cusp $0 \neq \infty$. Then there is a unique line $l = (\dots, v_{-1}, v_0, v_1, \dots)$ joining 0 to ∞. Let $L \subset P$ be the subgroup that preserves the line l. It is called a *Levi component* of P. Let $\overline{P}$ be the stabilizer of the cusp 0. Then $L = P \cap \overline{P}$. The line l intersects each horosphere in exactly one vertex.

Let $M \stackrel{\text{def}}{=} L \cap S$. Then M is the pointwise stabilizer of the line l. The group M is compact and normal in L and L/M is infinite cyclic. Therefore the exact sequence

$$1 \to M \to L \to C \to 1$$

splits. Here C denotes an infinite cyclic group. Fix a splitting $s : C \to L$ and let A denote the image of s. Then $L = AM = MA$. The group A is infinite cyclic and there is a unique generator a_1 with $a_1 v_k = v_{k-1}$ for every $k \in \mathbb{Z}$. Let

$$A^- = \{a_1, a_1^2, a_1^3, \dots\}$$

be the “negative Weyl chamber”.

An element g of G is called *elliptic* if it fixes a point of X. This point can be chosen to be a vertex or a midpoint of an edge. If g fixes a midpoint, then g^2 fixes both vertices of that particular edge. If g does not fix a point of X then it is called *hyperbolic*. Let G_{hyp} denote the set of hyperbolic elements of G. For $g \in G_{\text{hyp}}$ the minimum

$$l(g) \stackrel{\text{def}}{=} \min_{x \in X} d(gx, x)$$

is attained exactly on a line $l_g = (\dots, v_{-1}(g), v_0(g), v_1(g), \dots)$ which is preserved by g. Since the line l_g can be mapped to our given line by some $h \in G$, it follows that g is conjugate to an element of $L = AM$. It is not hard to see that every hyperbolic element of L is conjugate to a uniquely determined element of A^-.

Fix a finite ring R with q elements. At each vertex v there are q edges which do not point towards ∞. We label these edges with the elements of R in a way that

- for $v \in l$ the edge pointing towards the cusp 0 is labelled by $0 \in R$,
- the labelling is invariant under the action of A.

For each labelled edge e let $lab(e) \in R$ denote the label. For $b \in \partial X \setminus \{\infty\}$ there is a unique line $(\dots, w_{-1}, w_0, w_1, \dots)$ running from ∞ to b. Define $\psi(b)$ in the ring of formal Laurent series $R((t))$ by

$$\psi(b) \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} lab(\overline{w_j, w_{j+1}})\, t^j.$$

Note that for every b there is $N \in \mathbb{N}$ such that $lab(\overline{w_j, w_{j+1}}) = 0$ for every $j \leqslant -N$.

The map ψ is a bijection from $\partial X \setminus \{\infty\}$ to the ring $R((t))$.

Note that the action of S on $\partial X \setminus \{\infty\}$ defines an injection $\eta : S \hookrightarrow \mathrm{Per}(\partial X \setminus \{\infty\})$. We will now describe the image of this injection. Let $x, y \in \partial X \setminus \{\infty\}$ and assume $x \neq y$. The lines l_x, l_y joining x and y to ∞ meet at a vertex $v_{x,y}$. A permutation g on $\partial X \setminus \{\infty\}$ lies in $\eta(S)$ if and only if for any such pair x, y the vertices $v_{x,y}$ and $v_{gx,gy}$ lie in the same horosphere and for two pairs x, y and x', y' we have $v_{x,y} = v_{x',y'} \implies v_{gx,gy} = v_{gx',gy'}$.

Using this characterization it is easy to see that for every $n_R \in R((t))$ there is a $n \in S$ such that

$$n(b) = \psi^{-1}\big(\psi(b) + n_R\big).$$

Let $N \subset S$ be the subgroup of all n of this form. In other words, N consists of those elements of S which act by addition of an element of $R((t))$ on ∂X. The group N thus is isomorphic to the additive group of $R((t))$ and acts transitively on $\partial X \setminus \{\infty\}$. It follows that $S = MN$ and $P = MAN$. The group N is a closed subgroup of G. Indeed, it carries the t-adic topology of $R((t))$.

Let $\overline{P}$ denote the stabilizer of the cusp 0. Define $\overline{N}$ in an analogous fashion to N to get $\overline{P} = MA\overline{N}$.

There is an element $w \in G$ with $w^2 = 1$ and $wv_j = v_{-j}$ for every j. Further w can be chosen so that

$$waw = waw^{-1} = a^{-1}$$

for every $a \in A$. Then w is called *the non-trivial Weyl element* and $W = \{1, w\}$ is the *Weyl group* to A.

Lemma 2.1 *(Bruhat decomposition). The group G can be decomposed into disjoint sets,*

$$G = P \cup PwP.$$

Further, $PwP = NwP = PwN$.

Proof. The group N acts transitively on the cusps different from ∞. This implies that for every $g \in G \setminus P$ there is $n \in N$ with $ng\infty = 0$. This implies $wng \in P$ and so $g \in NwP$. □

3. Trace formula and duality

Let G be a totally disconnected locally compact group, so G has a basis of unit neighbourhoods consisting of compact open subgroups. By a *representation* of G we mean a group homomorphism $\pi : G \to \mathrm{GL}(V_\pi)$ for some complex vector space V_π. By abuse of notation we will write π for the space V_π as well. For every compact open subgroup K of G let π^K denote the space of K-fixed vectors. We say that the representation π is *smooth* if every $v \in \pi$ is fixed by some compact open subgroup K.

Let π be a smooth representation. For a compact subgroup K of G and $v \in \pi$ let

$$\mathcal{P}_K(v) \stackrel{\text{def}}{=} \frac{1}{\mathrm{vol}(K)} \int_K \pi(k)v\, dk,$$

the projection to the K-invariants. In other words, $\mathcal{P}_K$ is a linear map on π given as follows. For given $v \in \pi$ there exists a subgroup K' of finite index in K such that v is invariant under K'. Then

$$\mathcal{P}_K(v) = \frac{1}{|K/K'|} \sum_{x \in K/K'} \pi(x)v.$$

It is easy to see that π is the direct sum of the space π^K and the kernel $\pi[K]$ of the linear map $\mathcal{P}_K$.

A smooth representation π is *admissible* if for every compact open subgroup K the space π^K is finite dimensional.

Suppose that G admits a uniform lattice Γ, i.e., a discrete subgroup such that $\Gamma\backslash G$ is compact. Then G is unimodular [6, Ex. 14.2]. The space of $C^\infty(\Gamma\backslash G)$ locally constant functions on $\Gamma\backslash G$ carries an admissible representation R given by

$$R(y)\varphi(x) \stackrel{\text{def}}{=} \varphi(xy).$$

A function f on G is called a *Hecke function* if f is integrable and there is a compact open subgroup K of G such that f factors over $K\backslash G/K$. For any Hecke function f the integral

$$R(f) \stackrel{\text{def}}{=} \int_G f(x)\, R(x)\, \mathrm{d}x$$

defines a linear operator on $C^\infty(\Gamma\backslash G)$. For $g \in G$ and f a function on G define the *orbital integral*

$$\mathcal{O}_g(f) \stackrel{\text{def}}{=} \int_{G_g\backslash G} f\bigl(x^{-1}gx\bigr)\, \mathrm{d}x$$

whenever the integral exists. Here G_g is the centralizer of g in G. We have to fix a Haar measure on G_g here.

Proposition 3.1 *(Trace formula). Let f be a Hecke function. Then*

$$\operatorname{tr} R(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f),$$

where the sum on the right-hand side runs over the set of all conjugacy classes in the group Γ and Γ_γ denotes the centralizer of γ in Γ.

Proof. The proof is the same as that of the corresponding assertion for p-adic linear groups [5]. □

The trace formula can also be used to show the Ihara/Bass-identity (see [18,19]).

We now turn to duality. Let $\mathrm{Adm}(G)$ denote the category of all admissible smooth representations of G. We introduce the setup of topological (continuous) representations. Let $\mathcal{C}(G)$ denote the category of continuous representations of G on locally convex, complete, Hausdorff topological vector spaces. The morphisms in $\mathcal{C}(G)$ are continuous linear G-maps.

To any $V \in \mathcal{C}(G)$ we can form V^∞, the space of smooth vectors, which is by definition the space of all vectors in V which have an open stabilizer. By the continuity of the representation it follows that V^∞ is dense in V. We call the representation V *admissible* if V^∞ is admissible. Let $\mathrm{Adm}_{\mathrm{top}}(G)$ denote the category of all admissible topological representations. The functor:

$$\begin{aligned} F : \mathrm{Adm}_{\mathrm{top}}(G) &\to \mathrm{Adm}(G), \\ W &\mapsto W^\infty \end{aligned}$$

is easily seen to be exact.

Let V be in $\mathrm{Adm}(G)$ then any $W \in \mathrm{Adm}_{\mathrm{top}}(G)$ with $FW = V$ will be called a *completion* of V.

For $V \in \mathrm{Adm}(G)$ let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the dual module and let $\widetilde{V}$ be the space of smooth vectors in V^*. Then $\widetilde{V}$ is admissible again and the natural map from V to $\widetilde{\widetilde{V}}$ is an isomorphism.

On the space $C(G)$ of continuous maps from G to the complex numbers we have two actions of G, the left and the right action given by

$$L(g)\varphi(x) := \varphi\bigl(g^{-1}x\bigr), \qquad R(g)\varphi(x) := \varphi(xg),$$

where $g, x \in G$ and $\varphi \in C(G)$. For $V \in \mathrm{Adm}(G)$ let

$$V^{-\infty} := \mathrm{Hom}_G\bigl(\widetilde{V}, C(G)\bigr),$$

where we take G-homomorphisms with respect to the right action, so $V^{-\infty}$ is the space of all linear maps f from $\widetilde{V}$ to $C(G)$ such that $f(g.v^*) = R(g)f(v^*)$. Then $V^{-\infty}$ becomes a G-module via the left translation: for $\alpha \in V^{-\infty}$ we define $g.\alpha(v^*) := L(g)\alpha(v^*)$. We call $V^{-\infty}$ the *maximal completion* of V. The next lemma and the next proposition will justify this terminology. First observe that the topology of locally uniform convergence may be installed on $V^{-\infty}$ to make it an element of $\mathcal{C}(G)$. By $V \mapsto V^{-\infty}$ we then get a functor $R : \mathrm{Adm}(G) \to \mathrm{Adm}_{\mathrm{top}}(G)$.

Lemma 3.2. *We have* $(V^{-\infty})^\infty \cong V$.

Proof. There is a natural map $\Phi : V \to V^{-\infty}$ given by $\Phi(v)(v^*)(g) := v^*(g^{-1}.v)$, for $v^* \in \widetilde{V}$ and $g \in G$. This map is clearly injective. To check surjectivity let $\alpha \in (V^{-\infty})^\infty$, then the map $\widetilde{V} \to \mathbb{C}$, $v^* \mapsto \alpha(v^*)(1)$ lies in $\widetilde{\widetilde{V}}$. Since the natural map $V \to \widetilde{\widetilde{V}}$ is an isomorphism, there is a $v \in V$ such that $\alpha(v^*)(1) = v^*(v)$ for any $v^* \in \widetilde{V}$, hence $\alpha(v^*)(g) = \alpha(g.v^*)(1) = g.v^*(v) = v^*(g^{-1}.v) = \Phi(v)(v^*)(g)$, hence $\alpha = \Phi(v)$. □

It follows that the functor R maps $\mathrm{Adm}(G)$ to $\mathrm{Adm}_{\mathrm{top}}(G)$ and that $FR = \mathrm{Id}$.

Proposition 3.3. *The functor* $R : \mathrm{Adm}(G) \to \mathrm{Adm}_{\mathrm{top}}(G)$ *mapping* $V \in \mathrm{Adm}(G)$ *to its maximal completion* $V^{-\infty}$ *is right adjoint to the functor* $F : \mathrm{Adm}_{\mathrm{top}}(G) \to \mathrm{Adm}(G)$ *mapping* W *to its smooth part* W^∞. *So for* $W \in \mathrm{Adm}_{\mathrm{top}}(G)$ *and* $V \in \mathrm{Adm}(G)$ *we have a functorial isomorphism*:

$$\mathrm{Hom}_G(FW, V) \cong \mathrm{Hom}_{\mathcal{C}(G)}(W, RV).$$

Proof. We have a natural map

$$\mathrm{Hom}_{\mathcal{C}(G)}\big(W, V^{-\infty}\big) \to \mathrm{Hom}_G\big(W^\infty, V\big),$$
$$\alpha \mapsto \alpha|_{W^\infty}.$$

Since W^∞ is dense in W this map is injective. For surjectivity we first construct a map $\psi : W \to (W^\infty)^{-\infty}$. For this let W' be the topological dual of W and for $w' \in W'$ and $w \in W$ let

$$\psi_{w',w}(x) := w'\big(x^{-1}.w\big), \quad x \in G.$$

The map $w \mapsto \psi_{.,w}$ gives an injection

$$W \hookrightarrow \mathrm{Hom}_G\big(W', C(G)\big).$$

Let $\varphi \in \widetilde{FW} = \widetilde{W^\infty}$, then φ factors over $(FW)^K = (W^\infty)^K$ for some compact open subgroup $K \subset G$. But since $(FW)^K = W^K$ it follows that φ extends to W. The admissibility implies that φ is continuous there. So we get $\widetilde{FW} = \widetilde{W^\infty} \hookrightarrow W'$. The restriction then gives

$$\mathrm{Hom}_G\big(W', C(G)\big) \to \mathrm{Hom}_G\big(\widetilde{FW}, C(G)\big).$$

From this we get an injection

$$\psi : W \hookrightarrow (FW)^{-\infty} = \big(W^\infty\big)^{-\infty},$$

which is continuous. Let $\zeta : W^\infty \to V$ be a morphism in $\mathrm{Adm}(G)$. We get

$$\alpha(\zeta) : W \hookrightarrow (FW)^{-\infty} \xrightarrow{\zeta^{-\infty}} V^{-\infty},$$

with $\alpha(\zeta)|_{W^\infty} = \zeta$, i.e. the desired surjectivity. □

Let now Γ denote a cocompact torsion-free discrete subgroup of G and let $C(\Gamma\backslash G)$ denote the space of all continuous functions on $\Gamma\backslash G$.

The next theorem is a version of the classical duality theorem [7] adapted to our context.

Theorem 3.4 *(Duality Theorem). Let* Γ *be a cocompact torsion-free discrete subgroup of* G, *then for any* $V \in \mathrm{Adm}(G)$:

$$H^0\big(\Gamma, V^{-\infty}\big) \cong \mathrm{Hom}_G\big(C^\infty(\Gamma\backslash G), V\big)^*,$$

and both sides are finite dimensional $\mathbb{C}$*-vector spaces.*

There is a version for higher cohomologies in [5] but this will not be needed here.

Proof. Since $V^{-\infty} = \mathrm{Hom}_G(\widetilde{V}, C(G))$ it follows that

$$H^0(\Gamma, V^{-\infty}) = \mathrm{Hom}_G(\widetilde{V}, C(\Gamma\backslash G)) = \mathrm{Hom}_G(\widetilde{V}, C^\infty(\Gamma\backslash G)),$$

since any G-homomorphism from the smooth $\widetilde{V}$ to $C(\Gamma\backslash G)$ has image in $C^\infty(\Gamma\backslash G)$. Dualizing the right-hand side gives the claim. □

4. A character identity

The proofs in this section are inspired by [3,4].

Fix a line $l_0 = (\dots, v_{-1}, v_0, v_1, \dots)$. Let ∞ denote the cusp given by the ray $(v_0, v_1, \dots)$ and let 0 denote the cusp given by the ray $(v_0, v_{-1}, v_{-2}, \dots)$. Let P, L, M, S, A be the groups defined in Section 2. For $a \in A$ let $v(a)$ be defined by $a.v_j = v_{j+v(a)}$.

For $j \geqslant 0$ let K_j be the group of all $g \in G$ with $gv_k = v_k$ for $|k| \leqslant j$. Then K_0 is a maximal compact subgroup and the K_j are all open and compact. Let $N_j = N \cap K_j$ and $\overline{N}_j = \overline{N} \cap K_j$ and set $S_j = K_j \cap S$.

For any set X let $\mathbb{1}_X$ denote its characteristic function. For $f, g \in L^1(G)$ denote by $f * g$ their convolution product, so

$$f * g(x) = \int_G f(y) g(y^{-1}x)\,\mathrm{d}y.$$

Lemma 4.1. *Let $j \geqslant 1$.*

(a) *We have the Iwahori factorization, $K_j = \overline{N}_j M N_j = N_j M \overline{N}_j$.*
(b) *$S_j = MN_j = N_j M$.*
(c) *For $a \in A^-$,*
   - *$aN_j a^{-1} = N_{j+l(a)}$,*
   - *$a^{-1}\overline{N}_j a = \overline{N}_{j+l(a)}$.*

(d) *For $a, b \in A^-$,*

$$\mathbb{1}_{K_j a K_j} * \mathbb{1}_{K_j b K_j} \quad \textit{is a positive constant times } \mathbb{1}_{K_j ab K_j}.$$

Proof. For (a) note that if $k \in K_j$, $j \geqslant 1$, then $k0 \neq \infty$. hence there are $n_j \in N_j$ and $\bar{n}_j \in \overline{N}_j$ such that

$$\bar{n}_j k n_j \infty = \infty,$$
$$\bar{n}_j k n_j 0 = 0,$$

which implies $\bar{n}_j k n_j \in M$. Now the first part of (a) follows. The second is obtained by taking inverses. Item (b) follows similarly. Item (c) is immediate from the fact that $av_j = v_{j+v(a)}$ and N_j being the stabilizer of v_j in N.

Finally, we show (d). As sets one has $K_j a K_j \cdot K_j b K_j = K_j ab K_j$ because for $k_j = n_j m \bar{n}_j$ one has $ak_j b = an_j a^{-1} \cdot ab \cdot b^{-1} mb \cdot b^{-1} \bar{n}_j b$. This implies the claim. □

Let π be an admissible representation of G. For any subgroup H of G let $\pi[H]$ denote the complex vector space generated by all elements of the form $v - \pi(h)v$, $v \in \pi$, $h \in H$. Let π_H be the quotient $\pi/\pi[H]$. Then π_H is the largest quotient of π on which H acts trivially.

Let f be a compactly supported Hecke function on G and π be an admissible Banach representation of G. The operator-valued integral

$$\pi(f) \stackrel{\mathrm{def}}{=} \int_G f(x)\pi(x)\,\mathrm{d}x$$

is well defined and since f can be written as

$$f=\sum_{i=1}^{m} c_i \mathbb{1}_{Kg_iK}$$

for some compact open subgroup K of G, where $c_i\in\mathbb{C}$ and $g_i\in G$, the integral equals

$$\pi(f)=\sum_{i=1}^{m} c_i \int\limits_{Kg_iK} \pi(x)\,\mathrm{d}x=\sum_{i=1}^{m} c_i \operatorname{vol}(Kg_iK)\,\mathcal{P}_K\pi(g_i)\mathcal{P}_K.$$

The last line will serve as definition for $\pi(f)$ for π being any smooth representation. In the sequel, we will also write $\pi(KaK)$ for $\pi(\mathbb{1}_{KaK})=\int_{KaK}\pi(x)\,\mathrm{d}x$.

Corollary 4.2. *For $v\in\pi^{K_j}$ with image u in π_S the vector*

$$\frac{1}{\operatorname{vol}(K_jaK_j)}\pi(\mathbb{1}_{K_jaK_j})v$$

has image $\pi_S(a)u$.

Proof. This follows from Lemma 4.1. $\square$

Proposition 4.3. *Suppose π is finitely generated as a G-module. Then π_S is a finitely generated A-module.*

Proof. Let E be a finite generating set of π. Let $K\subset G$ be a compact open subgroup such that $E\subset\pi^K$. Let Λ be a finite subset of G such that $P\Lambda K=G$. Since π is the linear span of $\pi(G)E$ it follows that π_S is as an A-module generated by the image of $\pi(\Lambda)E$. $\square$

Proposition 4.4.

(a) *If $v\in\pi$ is fixed by $M\overline{N}_j$, then $\mathcal{P}_{K_j}(v)=\mathcal{P}_{N_j}(v)$. If v is fixed by $\overline{N}_j$, then $\mathcal{P}_{K_j}(v)=\mathcal{P}_{S_j}(v)$.*
(b) *The canonical projection from π^M to π_S is surjective.*
(c) *The canonical projection from π^{K_1} to π_S is surjective. In particular, π_S is finite dimensional.*

Proof. The first assertion is an easy consequence of the Iwahori decomposition. Let π_M be the largest quotient on which M acts trivially. Since M is compact, integration over M induces an isomorphism $\pi_M\to\pi^M$. Since the map $\pi_M\to\pi_S$ is surjective, the second claim follows.

For the third let U_S be a finite dimensional subspace of π_S and let U be a finite dimensional subspace of π^M mapping to U_S. There is $j\in\mathbb{N}$ such that $U\subset\pi^{M\overline{N}_j}$. Choose $a\in A$ such that $a^{-1}\overline{N}_1a\subset\overline{N}_j$. Then $\pi(a)U\subset\pi^{M\overline{N}_1}$. The decomposition $K_1=N_1M\overline{N}_1$ implies that the image of $\pi^{M\overline{N}_1}$ in π_S equals the image of π^{K_1}. So it follows that $\pi(a)U_S$ lies in the image of π^{K_1}. Hence its dimension, i.e., the dimension of U_S, is bounded by the dimension of π^{K_1}. We conclude that π_S is finite dimensional. Thus one can take U_S to be equal to π_S. But then $U_S=\pi(a)U_S$ lies in the image of π^{K_1}. $\square$

Proposition 4.5. *Let a_1 be the unique generator of A inside A^-. For every $j\geqslant 1$ there exists $n\in\mathbb{N}$ such that with $\pi_1^{K_j}\stackrel{\mathrm{def}}{=}\pi(\mathbb{1}_{K_ja_1^nK_j})\pi^{K_j}$,*

(a) *The projection from $\pi_1^{K_j}$ to π_S is a linear isomorphism.*
(b) *For each $m\geqslant 0$ the space $\pi_1^{K_j}$ is stable under $\pi(\mathbb{1}_{K_ja_1^mK_j})$.*

Proof. We need the following lemma.

Lemma 4.6. *For every compact subgroup H of G the space $\pi[H]$ coincides with the space of all $v \in \pi$ with $\int_H \pi(h)v\,\mathrm{d}h = 0$.*

Proof. For every $v \in \pi[H]$ we obviously have $\int_H \pi(h)v\,\mathrm{d}h = 0$. The other way round assume that $v \in \pi$ satisfies $\int_H \pi(h)v\,\mathrm{d}h = 0$. There exists a normal subgroup H' of finite index in H such that $v \in \pi^{H'}$. Let F denote the finite group H/H'. Then one has $0 = \sum_{x\in F} \pi(x)v$. Hence

$$v = \frac{1}{|F|}\sum_{x\in F}\bigl(v - \pi(x)v\bigr).$$

This shows that v lies in $\pi[H]$. □

Recall from Section 2 that S is the union of its compact open subgroups. Thus $\pi[S]$ is the union of the sets $\pi[U]$, when U runs over all compact open subgroups of S. Fix $j \in \mathbb{N}$. Choose a fixed compact open subgroup U of S such that $\pi[S] \cap \pi^{K_j} \subset \pi[U]$ and $S_j \stackrel{\mathrm{def}}{=} S \cap K_j \subset U$.

Lemma 4.7. *If $a_1^n U a_1^{-n} \subset S_j$ and $v \in \pi[S] \cap \pi^{K_j}$, then $\pi(\mathbb{1}_{K_j a_1^n K_j})v = 0$.*

Proof. The vector $\pi(\mathbb{1}_{K_j a_1^n K_j})v$ differs from $\mathcal{P}_{K_j}(\pi(a_1^n)v)$ only by a scalar. By Proposition 4.4 the latter equals $\mathcal{P}_{S_j}(\pi(a_1^n)v)$. But

$$\mathcal{P}_{S_j}\bigl(\pi\bigl(a_1^n\bigr)v\bigr) = (const)\int_{S_j} \pi(s)\pi\bigl(a_1^n\bigr)v\,\mathrm{d}s = (const)\pi\bigl(a_1^n\bigr)\int_{a_1^n S_j a_1^{-n}} \pi(s)v\,\mathrm{d}s = 0. \qquad \square$$

Choose n to be large enough that $a_1^n U a_1^{-n} \subset S_j$ and define $\pi_1^{K_j}$ to be $\pi(K_j a_1^n K_j)\pi^{K_j}$. We first show that the projection from $\pi_1^{K_j}$ to π_S is surjective. For this let $u \in \pi_S$. By Proposition 4.4 there is $v \in \pi^{K_j}$ whose image in π_S equals $\pi_S(a_1^{-n})u$. By Corollary 4.2, $\mathcal{P}_{K_j}(\pi(a_1^n)v)$ has image u.

For the injectivity let $v_0 \in \pi^{K_j}$ and assume that $v = \pi(K_j a_1^n K_j)v_0$ lies in $\pi[S]$. We have to show that $v = 0$.

By choice of U, $v \in \pi[U]$. Now v is also, up to a constant, equal to $\mathcal{P}_{K_j}(\pi(a_1^n)v_0) = \mathcal{P}_{N_j}(\pi(a_1^n)v_0)$. Therefore

$$\int_U \pi(u)v\,\mathrm{d}u = 0 = \int_U \pi(u)\,\mathrm{d}u \int_{N_j} \pi(n_j)\pi\bigl(a_1^n\bigr)v_0\,\mathrm{d}n_j = \int_U \pi(u)\pi\bigl(a_1^n\bigr)v_0\,\mathrm{d}u = \pi\bigl(a_1^n\bigr)\int_{a_1^{-n}Ua_1^n} \pi(u)v_0\,\mathrm{d}u.$$

So $v_0 \in \pi[S]$ and by Lemma 4.7 it follows $v = 0$.

To prove (b) note that the above is valid for every large enough n, so all the spaces $\pi(K_j a_1^n K_j)\pi^{K_j}$ have the same dimension. But for $m \geqslant 0$,

$$\pi\bigl(K_j a_1^m K_j\bigr)\pi\bigl(K_j a_1^n K_j\bigr)\pi^{K_j} = \pi\bigl(K_j a^{m+n} K_j\bigr)\pi^{K_j}.$$

The proposition follows. □

We now state the main theorem of this section.

Theorem 4.8. *For all $j, m \geqslant 1$,*

$$\mathrm{tr}\left(\frac{1}{\mathrm{vol}(K_j a_1^m K_j)}\pi(\mathbb{1}_{K_j a_1^m K_j})\right) = \mathrm{tr}\,\pi_S\bigl(a_1^m\bigr).$$

Proof. Since $(\mathbb{1}_{K_j a_1^m K_j})^n = c\mathbb{1}_{K_j a_1^{mn} K_j}$, the Proposition 4.5 implies that

$$\mathrm{tr}\left(\frac{1}{\mathrm{vol}(K_j a_1^m K_j)}\pi(\mathbb{1}_{K_j a_1^m K_j})\right) = \frac{1}{\mathrm{vol}(K_j a_1^m K_j)}\,\mathrm{tr}\bigl(\pi(\mathbb{1}_{K_j a_1^m K_j}) \mid \pi_1^{K_j}\bigr) = \mathrm{tr}\bigl(a_1^m|\pi_S\bigr). \qquad \square$$

Corollary 4.9. *Let π be an admissible representation. On G_{hyp} define a conjugation invariant function θ_π by $\theta_\pi(a) = \operatorname{tr}(a|\pi_S)$, $a \in A^-$. Then for every Hecke function f which is supported on G_{hyp} and biinvariant under M,*

$$\operatorname{tr}\pi(f) = \int_G f(x)\,\theta_\pi(x)\,\mathrm{d}x.$$

Proof. The claim follows from the theorem for Hecke functions of the form $f = \mathbb{1}_{K_j a K_j}$, $a \in A^-$. Every Hecke function with support in G_{hyp} is a linear combination of functions which are conjugate to functions of the latter form. □

5. The Lefschetz formula

Recall that G is the automorphism group of a $(q+1)$-regular tree. Note that such a group G admits a lattice Γ as above, hence G is unimodular. Let g be a hyperbolic element of G. Let $l(g) = l = (\dots, v_{-1}, v_0, v_1, \dots)$ be the line that is preserved by g. After renumbering we can achieve that $gv_j = v_{j-l(g)}$. Let b be the parallelity class of the ray $(v_0, v_1, \dots)$. Let P, S, M, A be defined as before. Then $g \in L = AM$. The centralizer G_g of g preserves l and therefore is a subgroup $G_g = L_g$ of L. Let $M_g = G_g \cap M$, then we have an exact sequence

$$1 \to M_g \to L_g \to C \to 1$$

for an infinite cyclic group C. Let $A_g \subset L_g$ be an image of a splitting $s : C \to L_g$, then $L_g = A_g M_g$. We normalize the Haar measure on the compact group M_g to have volume one and on A_g we choose $l(a_0)$ times the counting measure, where a_0 is a generator of A_g. Then the volume of $\langle g\rangle\backslash L_g$ equals $l(g)$.

Now suppose that Γ acts freely on X. Then every element $\gamma \in \Gamma$, $\gamma \neq 1$ is hyperbolic and the centralizer Γ_γ is cyclic. Let γ_0 be the unique generator of Γ_γ with $\gamma = \gamma_0^\mu$ with $\mu > 0$. Then

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) = l(\gamma_0).$$

Alternatively we call an element τ of Γ a *primitive* element if $\tau = \sigma^n$ for $\sigma \in \Gamma$ and $n \in \mathbb{N}$ implies $n = 1$. Then γ_0 is the unique primitive element such that $\gamma = \gamma_0^\mu$ with $\mu > 0$.

Proposition 5.1. *Assume that Γ acts freely on X. Then for every Hecke function f,*

$$\operatorname{tr} R(f) = \sum_{[\gamma]} l(\gamma_0)\,\mathcal{O}_\gamma(f).$$

Proof. This follows from the trace formula. □

Fix a cusp $b = (v_0, v_1, \dots)$ and a line $l = (\dots, v_{-1}, v_0, v_1, \dots)$ prolonging b. Let L, P, S, M, A be defined as before. Let L^- denote the set of all $g \in L$ such that $l(g) > 0$ and $gv_j = v_{j-l(g)}$, i.e., L^- is the set of elements of L that "move away" from b. Let $A^- = A \cap L$. Then $L^- = A^- M$. Let $\mathcal{E}(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in Γ such that γ is in G conjugate to an element $a_\gamma m_\gamma$ of $L^- = A^- M$.

Theorem 5.2 (Lefschetz formula). *Let $\Gamma \subset G$ be a discrete cocompact subgroup which acts freely on X. Then for every function φ on A^- such that $\sum_{a\in A^-} \varphi(a) q^{l(a)} < \infty$,*

$$\sum_{a\in A^-} \varphi(a)\, q^{l(a)} \operatorname{tr}(a|R_S) = \sum_{[\gamma]\in\mathcal{E}(\Gamma)} l(\gamma_0)\,\varphi(a_\gamma).$$

The proof will occupy the rest of the section. We start with the simple observation that the length map $l : G \to \mathbb{N}_0$ is continuous and hence $l^{-1}(n)$ is open for every $n \in \mathbb{N}_0$. An element $g \in G$ is elliptic iff $l(g) = 0$ and hyperbolic iff $l(g) > 0$.

We next need an integration formula. Fix the Haar measure on G that gives the stabilizer K_x of a vertex x the volume one. On A install the counting measure as Haar measure.

For $j \in \mathbb{Z}$ let $K(j)$ be the stabilizer of the point v_j and set $K = K(0) = K_0$. Define $N(j) = N \cap K(j)$. Then $N(j)$ is a compact open subgroup of N that stabilizes v_k for every $k \geqslant j$. The group N is the union of all the subgroups $N(j)$. We give N the Haar measure such that $N(0)$ gets volume one. Then $\mathrm{vol}(N(j)) = q^j$.

Lemma 5.3. *With the above normalizations we have for every $f \in L^1(G)$,*

$$\int_G f(x)\,\mathrm{d}x = \int_K \int_N \int_A f(kna)\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}k.$$

If f is supported in the open set of hyperbolic elements,

$$\int_G f(x)\,\mathrm{d}x = \int_K \int_N \int_{A^-} f\bigl(kn\,a(kn)^{-1}\bigr)\,q^{l(a)}\,\mathrm{d}a\,\mathrm{d}n.$$

For later use we also note that for $a \in A$, $f \in L^1(S)$,

$$\int_S f\bigl(asa^{-1}\bigr)\,\mathrm{d}s = q^{v(a)} \int_S f(s)\,\mathrm{d}s.$$

Here $v(a) \in \mathbb{Z}$ is defined by $a(v_j) = v_{j+v(a)}$. Then $a \in A^-$ is equivalent to $v(a) < 0$.

Proof. On $K\backslash G$ there is a unique G-invariant measure such that

$$\int_G f(x)\,\mathrm{d}x = \int_{K\backslash G} \int_K f(kx)\,\mathrm{d}k\,\mathrm{d}x.$$

The natural projection identifies NA with $K\backslash G$, so this measure gives a Haar measure on the group NA. But $\mathrm{d}n\,\mathrm{d}a$ also is a Haar measure, so the uniqueness of Haar measures gives the first assertion up to a scalar. Plugging in the test function $f = \mathbb{1}_K$ gives the first claim.

Recall that we write G_{hyp} for the set of all hyperbolic elements of G. Assume that f is supported in G_{hyp}. Note that the map

$$\begin{aligned} G/A \times A^- &\to G_{\mathrm{hyp}}, \\ (x, a) &\mapsto xax^{-1} \end{aligned}$$

is surjective and proper. Hence there is a nowhere vanishing function B on $G/A \times A^-$ such that

$$\int_G f(x)\,\mathrm{d}x = \int_{G/A} \int_{A^-} f\bigl(xax^{-1}\bigr)B(x, a)\,\mathrm{d}a\,\mathrm{d}x.$$

Since G is unimodular it follows that $\int_G f(yxy^{-1})\,\mathrm{d}x = \int_G f(x)\,\mathrm{d}x$ and hence B does not depend on x. Write $B(a)$ instead.

The projection $G \to G/A$ identifies KN with the quotient G/A. Via the first integral equality we get an identification of the measure on G/A with the measure $\mathrm{d}k\,\mathrm{d}n$. Thus,

$$\int_G f(x)\,\mathrm{d}x = \int_K \int_N \int_{A^-} f\bigl(kn\,a\,(kn)^{-1}\bigr)\,B(a)\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}k.$$

We are going to compute $B(a)$ by plugging in special test functions. First let $a_1 \in A^-$ be the unique element with $l(a_1) = 1$. For $k \in \mathbb{N}$ write $B(k)$ instead of $B(a_1^k)$. Fix some $a = a_1^k \in A^-$. Let f be the indicator function of the set $KaK \cap G_{\mathrm{hyp}}$. This function is K-central. The set KaK equals the set of all $g \in G$ that satisfy $d(gv_0, v_0) = k$. Since there are $(q+1)q^{k-1}$ vertices in distance k to v_0 it follows that

$$|KaK/K| = (q+1)q^{k-1}.$$

An element g of KaK is elliptic iff g fixes a point x half-way between v_0 and $g(v_0)$. Since for every point x in distance $k/2$ there are $q^{[k/2]}$ points in distance k that lie beyond x it follows that the volume of the set of elliptic elements in KaK equals $q^{[k/2]}$. Hence

$$\int_G f(x)\,\mathrm{d}x = (q+1)q^{k-1} - q^{[k/2]}.$$

For $a \in A$ recall that $v(a) \in \mathbb{Z}$ is defined by $a(v_j) = v_{j+v(a)}$. It is easy to see that

$$\{[a^{-1}, s]:\ s \in S_j\} = S_k,$$

where $k = \max(j, j - v(a))$. For $a \in A^-$ and $s \in S$ we have $d(sas^{-1}v_0, v_0) = d(as^{-1}v_0, s^{-1}v_0)$ and this number equals $l(a)$ if $s \in S_0$. If $s \in S_{j+1} \setminus S_j$ for some $j \geqslant 0$, then

$$d(as^{-1}v_0, s^{-1}v_0) = 2(j+1) + l(a).$$

Using $\mathrm{vol}(S_0) = 1$ and $\mathrm{vol}(S_{j+1} \setminus S_j) = q^j(q-1)$ we get

$$(q+1)q^{k-1} - q^{[k/2]} = \int_G f(x)\,\mathrm{d}x = B(k) + \sum_{\nu=1}^{[\frac{k-1}{2}]} B(k-2\nu)(q-1)q^{\nu-1}.$$

The unique solution to this recurrence formula is $B(k) = q^k$. This implies the second assertion of the lemma. For the third note that $\int_S f(asa^{-1})\,\mathrm{d}s$ is a Haar measure on S, therefore equals $c(a)\int_S f(s)\,\mathrm{d}s$ for a constant $c(a)$. Plugging in the test function $f = \mathbb{1}_K$ gives $c(a) = q^{v(a)}$. □

For $a \in A^-$ let C_a be the set of all kak^{-1}, where k ranges over K_1. This is an open subset of G. Define a function f on G by

$$f(kak^{-1}) \stackrel{\text{def}}{=} \frac{\varphi(a)}{\mathrm{vol}(C_a)}, \quad a \in A^-,\ k \in K_1$$

and $f(x) = 0$ if x is not of the form kak^{-1}. Then f is a Hecke function.

Lemma 5.4. *The function f is biinvariant under K_1. In particular, f is biinvariant under M.*

Proof. The set C_a equals the set of all $g \in G$ which are hyperbolic of length $l(a)$ with invariant line passing through v_{-1}, v_0, v_1. If g is in C_a and $k \in K_1$, then

- $d(gkx, x) \geqslant l(g)$ for every vertex x,
- $d(gkv_j, v_j) = d(gv_j, v_j) = l(g)$ for $j = -1, 0, 1$.

This implies that $gk \in C_a$. Since C_a is invariant under K_1-conjugation, it also follows that $kg \in C_a$. □

Let $a \in A^-$. Then

$$\mathcal{O}_a(f) = \varphi(a).$$

If we plug this function f into the trace formula we readily see that the geometric side will give the geometric side of the Lefschetz formula. For the spectral side we compute for an admissible representation π,

$$\mathrm{tr}\,\pi(f) = \int_G f(x)\theta_\pi(x)\,\mathrm{d}x = \int_{A^-} \varphi(a)q^{l(a)}\,\mathrm{tr}(a|\pi_S)\,\mathrm{d}a.$$

So Theorem 5.2 follows from the trace formula. □

6. Rationality of the zeta function

Let Y be a finite regular graph of valency, say, $q+1$, X its universal covering and G the automorphism group of X. Then there is a uniform torsion-free lattice Γ in G such that $Y = \Gamma\backslash X$. Recall the zeta function of Y,

$$Z_Y(u) = \prod_p \left(1 - u^{l(p)}\right)^{-1},$$

where the product runs over all prime loops in Y. Since Γ is the fundamental group of Y the set of prime loops is in bijection to the set of prime conjugacy classes $[\gamma_0]$ in Γ. Hence,

$$Z_Y(u) = Z_\Gamma(u) = \prod_{[\gamma_0]} \left(1 - u^{l(\gamma_0)}\right)^{-1}.$$

We compute the logarithmic derivative of Z_Y,

$$\frac{Z_Y'}{Z_Y}(u) = \sum_{[\gamma_0]} \frac{l(\gamma_0)u^{l(\gamma_0)-1}}{1-u^{l(\gamma_0)}} = \sum_{[\gamma_0]} l(\gamma_0) \sum_{n=1}^{\infty} \frac{u^{l(\gamma_0)n}}{u} = \sum_{[\gamma]} \frac{u^{l(\gamma)}}{u} l(\gamma_0).$$

Here the last sum runs over all nontrivial conjugacy classes in Γ and γ_0 is the underlying prime to γ. The last expression equals the geometric side of the Lefschetz formula for the test function

$$\varphi(a) \stackrel{\text{def}}{=} \frac{u^{l(a)}}{u},$$

which for $|u| < 1/q$ satisfies the condition of the Lefschetz formula. So in that range the function Z_Y'/Z_Y equals

$$\sum_{a \in A^-} \frac{u^{l(a)}}{u} q^{l(a)} \operatorname{tr}(a|R_S).$$

For every group homomorphism ψ from A to $\mathbb{C}^\times$ there is a unique number $\lambda \in \mathbb{C}^\times$ such that

$$\psi(a) = \lambda^{v(a)}.$$

For $\lambda \in \mathbb{C}^\times$ let $R_S(\lambda)$ be the *generalized λ-eigenspace*, i.e., $R_S(\lambda)$ is the largest A-submodule on which $(a - \lambda^{v(a)})$ acts nilpotently for every $a \in A$. Then

$$R_S = \bigoplus_\lambda R_S(\lambda).$$

Let m_λ denote the dimension of $R_S(\lambda)$. Then

$$\frac{Z_Y'}{Z_Y}(u) = \sum_\lambda m_\lambda \sum_{a\in A^-} \frac{u^{l(a)}}{u} q^{l(a)} \lambda^{-l(a)} = \sum_\lambda m_\lambda \sum_{n=1}^{\infty} \frac{(uq/\lambda)^n}{u} = \sum_\lambda m_\lambda \frac{q}{\lambda - uq} = -\sum_\lambda m_\lambda \frac{1}{u - \lambda/q}.$$

We have proved the following theorem.

Theorem 6.1. *The function $Z_Y(u)$ extends to a rational function on $\mathbb{C}$ with $Z_Y(u)^{-1}$ being a polynomial. The pole-order of Z_Y at $u = \lambda/q$, $\lambda \in \mathbb{C}^\times$, equals*

$$m_\lambda = \dim R_S(\lambda).$$

7. The Patterson conjecture

For $\lambda \in \mathbb{C}^\times$ let δ_λ be the character of $P = AS$ given by

$$\delta_\lambda(as) \stackrel{\text{def}}{=} \lambda^{v(a)}.$$

Let I_λ denote the representation of G induced from δ_λ. Then I_λ is the representation on the space of locally constant functions on G that satisfy

$$f(px) = \delta_\lambda(p) f(x)$$

for $p \in P$ and $x \in G$. The representation is given by

$$I_\lambda(y) f(x) = f(xy).$$

The representation I_λ is admissible.

Theorem 7.1. *The pole-order of $Z_Y(u)$ at $u = \lambda$ is*

$$\dim H^0\big(\Gamma, I_{1/\lambda q}^{-\infty}\big)$$

if $\lambda \neq \pm\sqrt{q}$ and

$$\dim H^0\big(\Gamma, \hat{I}_{\pm 1/q\sqrt{q}}^{-\infty}\big)$$

if $\lambda = \pm\sqrt{q}$, where $\hat{I}_{\pm 1/q\sqrt{q}}^{-\infty}$ is a certain self-extension of $I_{\pm 1/q\sqrt{q}}^{-\infty}$.

Corollary 7.2. *If $\lambda \notin \{\pm\sqrt{q}, \pm\frac{1}{q}, \pm 1\}$, then the pole order of $Z_Y(u)$ at $u = \lambda$ equals*

$$\dim H^0\big(\Gamma, I_\lambda^{-\infty}\big).$$

Proof. So far we know that the pole-order of $Z_Y(\lambda)$ equals $\dim R_S(\lambda q) = \dim C^\infty(\Gamma\backslash G)_S(\lambda q)$.

Lemma 7.3. *Let π be an irreducible admissible representation of G. If $\lambda \neq \pm\sqrt{q}$, then A acts semisimply on $\pi_S(\lambda)$. In any case the length of the A-module $\pi_S(\lambda)$ is at most 2.*

Proof. Let $\mathbb{C}_\lambda$ denote the one-dimensional P-module $\mathbb{C}$ on which P acts by δ_λ. For $\alpha \in \mathrm{Hom}(\pi, I_\lambda)$ let $\alpha(1) : \pi \to \mathbb{C}$ be given by

$$\alpha(1)(v) = \alpha(v)(1).$$

For $s \in S$ we get $\alpha(1)\pi(s)v = \alpha(1)v$ and so $\alpha(1)$ factors over π_S. In this way we get an isomorphism

$$\mathrm{Hom}_G(\pi, I_\lambda) \;\cong\; \mathrm{Hom}_A(\pi_S, \mathbb{C}_\lambda).$$

So, if π_S is not zero, then π, being irreducible, will inject into one I_λ and so π_S will inject into $I_{\lambda,S}$. So it suffices to show the assertion for $\pi = I_\lambda$.

The map $\psi : \pi = I_\lambda \to \mathbb{C}_\lambda$ given by $f \mapsto f(1)$ is a P-homomorphism. The group A acts on the image by δ_λ and this defines an irreducible quotient of π_S. We will show that the kernel of ψ gives an irreducible subrepresentation of π_S on which A acts by $\delta_{q/\lambda}$.

Recall the Bruhat decomposition $G = P \cup PwN$. Let f be in the kernel of ψ. Then f vanishes in a neighbourhood of 1 and thus the function $s \mapsto f(ws)$ is compactly supported on S, so the integral $\int_S f(ws)\,\mathrm{d}s$ converges. This integral defines an isomorphism $\ker(\psi)_S \to \mathbb{C}$. The group A acts as

$$\int_S f(wsa)\,\mathrm{d}s = q^{v(a)} \int_S f(was)\,\mathrm{d}s = q^{v(a)} \int_S f\big(a^{-1}ws\big)\,\mathrm{d}s = (q/\lambda)^{v(a)} \int_S f(ws)\,\mathrm{d}s.$$

The lemma follows. $\square$

To be able to apply Lemma 7.3 we have to show that $C^\infty(\Gamma\backslash G)$ is the direct sum

$$C^\infty(\Gamma\backslash G) = \bigoplus_{\pi \in \widehat{G}_{\mathrm{adm}}} N_\Gamma(\pi)\,\pi$$

of irreducible representations with finite multiplicities. This follows readily from the corresponding assertion for $L^2(\Gamma\backslash G)$. So suppose $\lambda \neq \pm\sqrt{q}$. Then by semisimplicity,

$$C^\infty(\Gamma\backslash G)_S(\lambda q) = H^0\big(A, C^\infty(\Gamma\backslash G)_S \otimes \mathbb{C}_{1/\lambda q}\big).$$

Since $A \cong \mathbb{Z}$ we get

$$\begin{aligned}\dim H^0\bigl(A, C^\infty(\Gamma\backslash G)_S \otimes \mathbb{C}_{1/\lambda q}\bigr) &= \dim \operatorname{Hom}_A\bigl(C^\infty(\Gamma\backslash G)_S, \mathbb{C}_{1/\lambda q}\bigr)\\ &= \dim \operatorname{Hom}_G\bigl(C^\infty(\Gamma\backslash G), I_{1/\lambda q}\bigr)\\ &= \dim H^0\bigl(\Gamma, I_{1/\lambda q}^{-\infty}\bigr).\end{aligned}$$

In the last equation we have used the duality theorem. This proves the first assertion of the theorem. For the second replace A with $A_2 = v^{-1}(2\mathbb{Z})$ and P by the finite index subgroup A_2S and follow the proof above.

For the corollary use the functional equation

$$Z_Y\left(\frac{1}{qu}\right) = \left(\frac{1-u^2}{q^2u^2-1}\right)^{r_1-r_0} q^{2r_1-r_0}\, u^{2r_1}\, Z_Y(u),$$

which is proved in [1, Corollary 3.10]. Here r_0 and r_1 are the number of vertices and edges respectively. □

Finally, we give the sheaf-theoretic version of Theorem 7.1. For $\lambda \in \mathbb{C}^\times$, $\lambda \neq \pm\sqrt{q}$, let

$$\mathcal{F}_\lambda \stackrel{\text{def}}{=} \Gamma\backslash\bigl(X \times I_{1/\lambda q}^{-\infty}\bigr),$$

where Γ acts diagonally. If $\lambda = \pm\sqrt{q}$ then replace I by $\hat{I}$ accordingly. The projection onto the first factor $\mathcal{F}_\lambda \to \Gamma\backslash X = Y$ makes $\mathcal{F}_\lambda$ a locally constant sheaf over Y. Since for a locally constant sheaf the sheaf-cohomology coincides with the group cohomology of the fundamental group, we get the following corollary.

Corollary 7.4. *The pole order of $Z_Y(u)$ at $u = \lambda$ equals the dimension of the space of global sections of the sheaf $\mathcal{F}_\lambda$.*

Appendix A. Zeta functions for c-sheaves

In this appendix we suggest a generalization of the Ihara/Bass zeta function.

As is well known, the zeta function can be twisted by a finite dimensional representation of the fundamental group Γ. The latter amounts to the same as the choice of a locally constant sheaf of vector spaces over the graph. In this appendix we show how the theory can be extended to a certain class of constructible sheaves.

A constructible sheaf on a stratified space $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ is a sheaf which is locally constant on each stratum $X_j \setminus X_{j-1}$. A graph X, viewed as a one-dimensional CW-complex, has a natural two-step stratification $X = X_1 \supset X_0$, where X_0 is the set of vertices, or the 0-skeleton. Let $\mathcal{F}$ be a constructible sheaf of Abelian groups on X.

For each edge s let x_e be its midpoint and let $\mathcal{F}_e$ denote the stalk at x_e. Let v be an endpoint of e. Let $\mathcal{F}_v$ be the stalk at v. Recall that

$$\mathcal{F}_v = \varinjlim_{U \ni v} \mathcal{F}(U),$$

where the limit runs over all open neighbourhoods of v. Let $s \in \mathcal{F}_v$. Then there is an open neighbourhood U of v and a section $s_U \in \mathcal{F}(U)$ such that s is the class of s_U in $\mathcal{F}_v$. Since U is open there is a point $y \in U \cap e$, where e here denotes the open edge. So s_U also defines a point in the stalk $\mathcal{F}_y$ over y. Since $\mathcal{F}$ is constant on the contractible space e there is a canonical isomorphism $\mathcal{F}_y \cong \mathcal{F}_e$, so s_U defines an element of $\mathcal{F}_e$. Since $\mathcal{F}$ is constant on e this element does not depend on the choices of y or s_U, but only on s. Thus we get a map

$$\varphi_v^e : \mathcal{F}_v \to \mathcal{F}_e.$$

It is easy to see that the sheaf $\mathcal{F}$ can be recovered up to isomorphism from the stalks $\mathcal{F}_v$, $\mathcal{F}_e$ and the maps φ_v^e. So a constructible sheaf on a graph is given by the following data

- an Abelian group $\mathcal{F}_v$ for every vertex v,
- an Abelian group $\mathcal{F}_e$ for every edge e, and
- a group homomorphism $\varphi_v^e : \mathcal{F}_v \to \mathcal{F}_e$ whenever v is an endpoint of e.

A *cosheaf* is the dual construction to a sheaf, so a constructible cosheaf is given by Abelian groups $\mathcal{F}_v, \mathcal{F}_e$ as above and group homomorphisms

$$\psi_e^v : \mathcal{F}_e \to \mathcal{F}_v$$

whenever v is an endpoint of e. Instead of sheaves of Abelian groups one can also consider sheaves of vector spaces, general groups, etc.

An example for a cosheaf of groups on a graph X is given as follows. Let G be the automorphism group of X. For each vertex v let G_v be its stabilizer in G. For each edge e let G_e be its pointwise stabilizer in G. If v is an endpoint of e then G_e is a subgroup of G_v, so the injection $\psi_e^v : G_e \to G_v$ defines a constructible cosheaf of groups on X.

Next, let (ρ, V) be a representation of G. For each vertex v let $\mathcal{V}_v$ be the space of fixed points $\mathcal{V}_v = V^{G_v}$. Likewise for edges. If v is an endpoint of e then $\mathcal{V}_v$ is a subspace of $\mathcal{V}_e$, so the inclusion $\varphi_v^e : \mathcal{V}_v \hookrightarrow \mathcal{V}_e$ defines a sheaf $\mathcal{V}$ of vector spaces over X.

Assume the representation ρ to be smooth. Then we also have a cosheaf structure on $\mathcal{V}$ given by

$$\begin{aligned}\psi_e^v &: \mathcal{V}_e \to \mathcal{V}_v,\\ &v \mapsto \frac{1}{\operatorname{vol}(G_v)} \int_{G_v} \rho(x) v \, \mathrm{d}x.\end{aligned}$$

In this case we have $\psi_e^v \varphi_v^e = \mathrm{Id}_{\mathcal{V}_v}$. This motivates the following definition.

By a *c-sheaf* we mean a constructible sheaf $\mathcal{V}$ of finite dimensional complex vector spaces that also carries the structure of a constructible cosheaf such that $\psi_e^v \varphi_v^e = \mathrm{Id}_{\mathcal{V}_v}$ for every edge e and every endpoint v of e.

Let $\mathcal{V}$ be a c-sheaf on X. Let u, v be two adjacent vertices and e their connecting edge. Define $T_u^v = \psi_e^v \varphi_u^e$, then T_u^v maps $\mathcal{V}_u$ linearly to $\mathcal{V}_v$. For a path $p = (v_0, \dots, v_n)$ let $T_p : \mathcal{V}_{v_0} \to \mathcal{V}_{v_n}$ be given by

$$T_p = T_{v_{n-1}}^{v_n} \cdots T_{v_1}^{v_2} T_{v_0}^{v_1}.$$

Let c be a loop in X given by the path $(v_0, \dots, v_n)$. Consider the operator

$$T_c \stackrel{\text{def}}{=} T_{v_{n-1}}^{v_n} \cdots T_{v_0}^{v_1}$$

on $\mathcal{V}_0$. Note that this operator depends on the choice of the underlying path. However, for every $j \in \mathbb{N}$ the expression $\operatorname{tr}(T_c^j)$ does not depend on the choice of a path. Therefore also

$$\det\bigl(1 - u^{l(c)} T_c\bigr) = \exp\left(- \sum_{j=1}^{\infty} \frac{u^{jl(c)}}{j} \operatorname{tr} T_c^j \right)$$

does only depend on the loop c.

Theorem A.1. *Let X be a finite graph and $\mathcal{V}$ a c-sheaf on X. Then the infinite product*

$$Z_{\mathcal{V}}(u) \stackrel{\text{def}}{=} \prod_p \det\bigl(1 - u^{l(p)} T_p\bigr)^{-1},$$

which runs over all prime loops in X, converges for u small and extends to a rational function in u. The inverse $Z_{\mathcal{V}}(u)^{-1}$ is a polynomial.

Proof. First note that

$$\operatorname{tr}\bigl(T_{v_{j+n}}^{v_{j+n+1}} \cdots T_{v_j}^{v_{j+1}}\bigr) = \operatorname{tr}\bigl(\psi_{e_{j+1}}^{v_{j+n+1}} \varphi_{v_{j+n}}^{e_{j+1}} \cdots \psi_{e_j}^{v_{j+1}} \varphi_{v_j}^{e_j}\bigr) = \operatorname{tr}\bigl(\varphi_{v_j}^{e_j} \psi_{e_{j+1}}^{v_{j+n+1}} \varphi_{v_{j+n}}^{e_{j+1}} \cdots \psi_{e_j}^{v_{j+1}}\bigr).$$

Next consider the set $OE = OE(X)$ of oriented edges. For each oriented edge e let e^{-1} denote the same edge with reversed orientation and let $\bar{e}$ be the underlying simple edge. Define $\mathcal{V}_e$ to be a copy of $\mathcal{V}_{\bar{e}}$ and let

$$T : \bigoplus_{e \in OE} \mathcal{V}_e \to \bigoplus_{e \in OE} \mathcal{V}_e$$

be defined by

$$T(s) = \sum_{e'} \varphi_v^{e'} \psi_e^v(s),$$

where $s \in \mathcal{V}_e$ and the sum runs over all oriented edges $e' \neq e^{-1}$ such that the target point of e equals the start point of e'. As in [1], it emerges that

$$\operatorname{tr} T^n = \sum_{c:l(c)=n} l(c_0) \operatorname{tr} T_c.$$

Therefore,

$$\det(1-uT) = \exp\left(-\sum_{n=1}^{\infty} \frac{u^n}{n} \operatorname{tr} T^n\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{l(c)=n} l(c_0) \operatorname{tr} T_c\right)$$
$$= \exp\left(-\sum_{c_0} \sum_{m=1}^{\infty} \frac{u^{ml(c_0)}}{m} \operatorname{tr} T_{c_0}^m\right) = \prod_{c_0} \left(1 - u^{l(c_0)} T_{c_0}\right).$$

This finishes the proof. □

Appendix B. Graphs with infinite ends

Linear algebraic groups over local fields of positive characteristic may contain arithmetic subgroups Γ which are of finite covolume but not cocompact. The quotient $\Gamma\backslash X$ of the Bruhat–Tits building X then is infinite. The finite covolume condition amounts to the following. For each vertex v of X let Γ_v be its stabilizer in Γ. Then Γ_v is a finite group. It follows that

$$\sum_v \frac{1}{|\Gamma_v|} < \infty.$$

If the building is a tree then the graph $\Gamma\backslash X$ has the structure described below [15].

Let Y be a connected graph. Since it does not affect the zeta function we may assume that Y has no finite ends, i.e., no vertices of valency one. Next we assume the Y decomposes as

$$Y = Y_c \cup S_1 \cup \cdots \cup S_N,$$

where Y_c is finite (compact core) and each S_j is a *cusp sector*, i.e., the vertices in S_j are $v_0, v_1, v_2 \ldots,$ each v_j for $j \geqslant 1$ has valency 2, being adjacent exactly to v_{j-1} and v_{j+1}. The intersection of Y_c and S_j is $\{v_0\}$.

This decomposition is not uniquely determined and the zeta function given below will depend on the choice. One can, however, make the choice canonical by insisting, for example, that each end-vertex v_0 of a cusp sector has valency greater that 2. We will now define the zeta function for such a graph in analogy with the Selberg zeta function for noncompact arithmetic quotients of the upper half plane.

For each vertex v let $d(v)$ denote the distance of v to Y_c. We say that a path $(v_0, \ldots, v_n)$ is *essentially backtracking-free* if, whenever $v_{j-1} = v_{j+1}$, then $d(v_{j-1}) < d(v_j)$. This means that the only backtracking happens in the cusp sectors where one walks into the cusp sector and turns back once to walk back into the core again. Next a closed path is *essentially tail-less* if it either is indeed tail-less or if $d(v_0) > d(v_1)$, i.e., the path starts in a cusp sector. An *S-loop* is an equivalence class of essentially backtracking-free, essentially tail-less, closed paths under index shift as before. An S-loop is called *prime* if it is not a power of a shorter one.

As before let OE denote the set of oriented edges. We choose a weight $w : OE \to (0, \infty)$ such that

$$\sum_{e \in OE} w(e) < \infty.$$

This is the finite volume condition. For each S-loop c given by the oriented edges $(e_1, \ldots, e_n)$ let

$$w(c) \stackrel{\text{def}}{=} w(e_1) \cdots w(e_n).$$

Then $w(c^j) = w(c)^j$ for every $j \in \mathbb{N}$.

Theorem B.1. *The infinite product over all prime S-loops,*

$$Z_Y(u) \stackrel{\text{def}}{=} \prod_p \bigl(1 - u^{l(p)} w(p)\bigr)^{-1}$$

converges for u small and extends to a meromorphic function on $\mathbb{C}$. The inverse $Z_Y(u)^{-1}$ is entire.

Proof. Let $\ell^2(OE, w)$ be the Hilbert space of all functions $f : OE \to \mathbb{C}$ with

$$\sum_{e \in OE} \bigl|f(e)\bigr|^2 w(e) < \infty. \qquad \square$$

On $\ell^2(OE, w)$ define a linear operator T by

$$Tf(e) \stackrel{\text{def}}{=} \sum_{e'} f(e')\, w(e),$$

where the sum runs over all oriented edges e' such that the starting edge of e' equals the target edge of e, and $e' \neq e^{-1}$ except in the case when e lies in a cusp sector and points outward, in which case $e' = e^{-1}$ is allowed.

Lemma B.1. *The operator T is of trace class and for every $j \in \mathbb{N}$ we have*

$$\operatorname{tr} T^j = \sum_{l(c)=j} l(c_0)\, w(c),$$

where the sum runs over all S-loops c of length j and c_0 is the underlying prime S-loop to c.

Proof. Once we know that T is of trace class the formula for the trace is clear. To see that T is of trace class let e be an oriented edge and define $\varphi_e = \sum_{e'} \delta_{e'}$, where $\delta_{e'}$ is the delta function at e' and the sum runs over all $e' \in OE$ with $\text{start}(e') = \text{target}(e)$ and $e' \neq e^{-1}$ except if e is in a cusp sector and pointing outward in which case $e' = e^{-1}$ is allowed. Then we have for $f \in \ell(OE, w)$,

$$Tf = \sum_{e \in OE} \langle f, \varphi_e \rangle\, \delta_e.$$

To show that T is of trace class it therefore suffices to show that

$$\sum_{e \in OE} \|\varphi_e\| \|\delta_e\| < \infty.$$

Now $\|\delta_e\|^2 = w(e)$ and $\|\varphi_e\|^2 = \sum_{e'} w(e')$, from which it emerges that $\sum_e \|\delta_e\|^2 < \infty$ and $\sum_e \|\varphi_e\|^2 < \infty$, which implies the claim. The lemma follows. $\square$

Let $\lambda_1, \lambda_2, \ldots$ be the non-zero eigenvalues of T, each repeated according to its multiplicity. Then $\sum_j |\lambda_j| < \infty$ and so the product

$$\det(1 - uT) = \prod_j (1 - u\lambda_j)$$

converges for every $u \in \mathbb{C}$ and defines an entire function with zeros at the numbers λ_j^{-1}, $j \in \mathbb{N}$. Using the lemma, the same computation as in the last section yields

$$\det(1 - uT) = Z_Y(u)^{-1}. \qquad \square$$

References

[1] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (6) (1992) 717–797.
[2] U. Bunke, M. Olbrich, Γ-cohomology and the Selberg zeta function, J. Reine Angew. Math. 467 (1995) 199–219.
[3] W. Casselman, Characters and Jacquet modules, Math. Ann. 230 (1977) 101–105.

[4] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, unpublished. You can download it under the name "The p-adic notes" from http://www.math.ubc.ca/people/faculty/cass/research.html.
[5] A. Deitmar, Geometric zeta-functions on p-adic groups, Math. Japon. 47 (1) (1998) 1–17.
[6] J. Dieudonné, Treatise on Analysis, Academic Press, New York, 1976.
[7] I.M. Gelfand, M.I. Graev, I.I. Pyatetskii-Shapiro, Representation Theory and Automorphic Functions, Saunders, Philadelphia, PA, 1969.
[8] K. Hashimoto, Artin type L-functions and the density theorem for prime cycles on finite graphs, Internat. J. Math. 3 (6) (1992) 809–826.
[9] K. Hashimoto, On zeta and L-functions on finite graphs, Internat. J. Math. 1 (4) (1990) 381–396.
[10] K. Hashimoto, Zeta functions of finite graphs and representations of p-adic groups, in: Automorphic Forms and Geometry of Arithmetic Varieties, in: Adv. Stud. Pure Math., vol. 15, 1989, pp. 211–280.
[11] Y. Ihara, Discrete subgroups of PL(2, $k_{\wp}$), in: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., Boulder, CO, 1965, American Mathematical Society, Providence, RI, 1966, pp. 272–278.
[12] Y. Ihara, Algebraic curves mod p and arithmetic groups, in: Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., Boulder, CO, 1965, American Mathematical Society, Providence, RI, 1966, pp. 265–271.
[13] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[14] Y. Ihara, Non-Abelian class fields over function fields in special cases, in: Actes du Congrès International des Mathématiciens, Nice, 1970, Tome 1, Gauthier-Villars, Paris, 1971, pp. 381–389.
[15] J.-P. Serre, Trees, Springer, Berlin, 1980.
[16] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1) (1996) 124–165.
[17] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings. II, Adv. Math. 154 (1) (2000) 132–195.
[18] A. Terras, Fourier analysis on finite groups and applications, London Mathematical Society Student Texts, vol. 43, Cambridge University Press, Cambridge, 1999.
[19] A.B. Venkov, A.M. Nikitin, The Selberg trace formula, Ramanujan graphs and some problems in mathematical physics, Algebra i Analiz 5 (3) (1993) 1–76. Translation in St. Petersburg Math. J. 5 (3) (1994) 419–484.